Limit with shifted harmonic numbers.
https://www.linkedin.com/feed/update/urn:li:activity:6576008322640683009
Let $a \in (0,\infty)$. Calculate $\lim_{n\to\infty}\left(e^{\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n}} - e^{\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}}\right)$.

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First we have to do some preparations.
For any real $a > 0$ let $h_n(a) := \sum_{k=1}^{n}\frac{1}{a+k}, n \in \mathbb{N}$. Note that in particular for $a = 1$ we have $h_n(1) = H_{n+1}$, where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is n-th harmonic number.
Consider two sequences $l_n := h_n(a) - \ln(a+n+1)$ and $u_n := h_n(a) - \ln(a+n)$.
Note that $l_n < u_n, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty}(u_n - l_n) = \lim_{n\to\infty}\ln\left(1+\frac{1}{a+n}\right) = 0$.
Also note that $l_n < l_{n+1}, \forall n \in \mathbb{N}$ and $u_n > u_{n+1}, \forall n \in \mathbb{N}$.
Indeed, since $\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$ for any $x > 0$ then for any $n \in \mathbb{N}$ holds

$$l_{n+1} - l_n = \frac{1}{a+n+1} - \ln\left(1+\frac{1}{a+n+1}\right) > 0 \text{ and}$$
$$u_n - u_{n+1} = \ln\left(1+\frac{1}{a+n}\right) - \frac{1}{a+n+1} > 0.$$

Since $l_1 \le l_n < u_n \le u_1$ then convergent both sequences (l_n) and (u_n) and $\lim_{n\to\infty}(u_n - l_n) = 0$ implies that both have the same limit.
Let $\gamma(a) := \lim_{n\to\infty}(h_n(a) - \ln(a+n+1)) = \lim_{n\to\infty}(h_n(a) - \ln(a+n))$. Since (l_n) strictly increase and (u_n) strictly decrease then

$$h_n(a) - \ln(a+n+1) < \gamma(a) < h_n(a) - \ln(a+n) \Leftrightarrow$$
$$\ln(a+n) + \gamma(a) < h_n(a) < \ln(a+n+1) + \gamma(a), \forall n \in \mathbb{N}.$$

Noting that $h_{n+1}(a) = h_n(a+1) + \frac{1}{a}$, $\gamma(a) = \lim_{n\to\infty}(h_{n+1}(a) - \ln(a+n+1))$ and $\gamma(a+1) = \lim_{n\to\infty}(h_n(a+1) - \ln(a+1+n))$ we obtain $\gamma(a) = \gamma(a+1) + \frac{1}{a}$.
Since $\lim_{n\to\infty}(H_n - \ln n) = \lim_{n\to\infty}(H_n - \ln(n+1)) = \gamma$, where $\gamma \approx 0.5772$ is Euler's constant then $\gamma(1) = \lim_{n\to\infty}(h_n(1) - \ln(n+2)) = \lim_{n\to\infty}(H_{n+1} - \ln(n+2)) = \gamma$
Thus $\gamma(a)$ completely defined by functional equation $\gamma(a+1) = \gamma(a) - \frac{1}{a}, \forall a > 0$ and $\gamma(1) = \gamma$.
Since $-\gamma(a+1) = -\gamma(a) + \frac{1}{a}$ and digamma function $\psi(a)$ can be defined by functional equation $\psi(a+1) = \psi(a) + \frac{1}{a}$ and $\psi(1) = -\gamma$ then $\gamma(a) = -\psi(a)$.
Now we ready to solve the problem.
Let $L(a) := \lim_{n\to\infty}(e^{h_n(a)} - e^{h_{n-1}(a)})$. We have $L(a) = \lim_{n\to\infty}e^{h_{n-1}(a)}\left(e^{\frac{1}{a+n}} - 1\right) =$
$\lim_{n\to\infty}\frac{e^{h_{n-1}(a)}}{a+n}\cdot\frac{e^{\frac{1}{a+n}} - 1}{\frac{1}{a+n}} = \lim_{n\to\infty}\frac{e^{h_{n-1}(a)}}{a+n}$. Since

$$\ln(a+n-1) + \gamma(a) < h_{n-1}(a) < \ln(a+n) + \gamma(a) \Leftrightarrow$$
$$\frac{a+n-1}{a+n}e^{\gamma(a)} < \frac{e^{h_{n-1}(a)}}{a+n} < e^{\gamma(a)} \text{ then by Squeeze Principle}$$
$$L(a) = e^{\gamma(a)} = e^{-\psi(a)}$$